Two-level systems – classical resonance

Note Title 2/12/2005

The physics of all 2-level systems is the same. In fact, the physics of all 2-level systems is equivalent to the physics of a spin-½ particle!

Consider a 2-level system, with eigenstates $|1\rangle$ and $|2\rangle$ and energies E_1 and E_2

Choose the new energy origin as $\frac{1}{2}(E_1+E_2)$. The most general Hermitian Hamiltonian is:

$$H = \begin{pmatrix} \frac{1}{2}(E_1-E_2) & H_{12} \\ H_{12}^* & -\frac{1}{2}(E_1+E_2) \end{pmatrix}$$

$$= -\frac{\mu}{\hbar}\begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$$

complex number – should have real and imaginary parts

So, we associate:

$$B_z = -\frac{\hbar}{\mu}\,\tfrac{1}{2}(E_1 - E_2)$$

axial + longitudinal field

$$B_x = -\frac{\hbar}{\mu}\,\mathrm{Re}(H_{12})$$

$$B_y = \frac{\hbar}{\mu}\,\mathrm{Im}(H_{12})$$

Voila! This ficticious / pseudo-spin is the essence of the Bloch vector (beware of alternative nomeclature)!

People often use this notation:

if $|\psi\rangle = a|1\rangle + b|2\rangle$, then the pseudo-spin components are:

$$S_1 = \langle\psi|\sigma_x|\psi\rangle = a^*b + ab^*$$
$$S_2 = \langle\psi|\sigma_y|\psi\rangle = -i(a^*b - ab^*)$$
$$S_3 = \langle\psi|\sigma_z|\psi\rangle = |a|^2 - |b|^2$$

note the lack of units — this is a normalized spin vector

Or, if $|\psi\rangle = \cos\frac{\theta}{2}|1\rangle + e^{i\varphi}\sin\theta/2\,|2\rangle$

$$S_x = \hbar/2 \sin\theta\cos\varphi$$
$$S_y = \hbar/2 \sin\theta\sin\varphi$$
$$S_z = \hbar/2 \cos\theta$$

units are plugged in for a real spin - people do this both ways

So, we might as well always talk about two-level systems in terms of a spin-1/2 pseudospin.

Now, how do we deal with a time-dependent driving field? The classical magnetic resonance problem is very useful, since we should expect that it will get expectation values (like $\langle S_x\rangle$, $\langle S_y\rangle$, and $\langle S_z\rangle$) correct.

Note that this can only go so far!

example: $|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$

- classically, $S_x = \frac{\hbar}{2}$ $S_z = 0$ exactly
- quantum, $\langle S_x \rangle = \hbar/2$ $\langle S_z \rangle = 0$

but! $\langle \Delta S_z \rangle = \sqrt{\langle S_z^2 \rangle - \langle S_z \rangle^2} = \hbar/2$

classical analogy can't get this rms spread in repeated measurements right

Anyway, classical magnetic resonance:

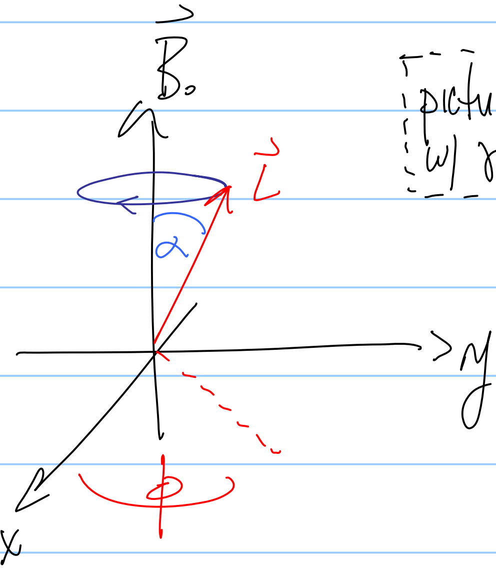
angular momentum $\vec{L}$

$\vec{\mu} = \gamma \vec{L}$ magnetic moment (γ = gyromagnetic ratio)

place it in a static field $\vec{B}_0$, torque is $\vec{\mu} \times \vec{B}_0$

classical equation of motion: $\dot{\vec{L}} = \vec{\mu} \times \vec{B}_0 \rightarrow \dot{\vec{L}} = \gamma \vec{L} \times \vec{B}_0$

just get Larmor precession:



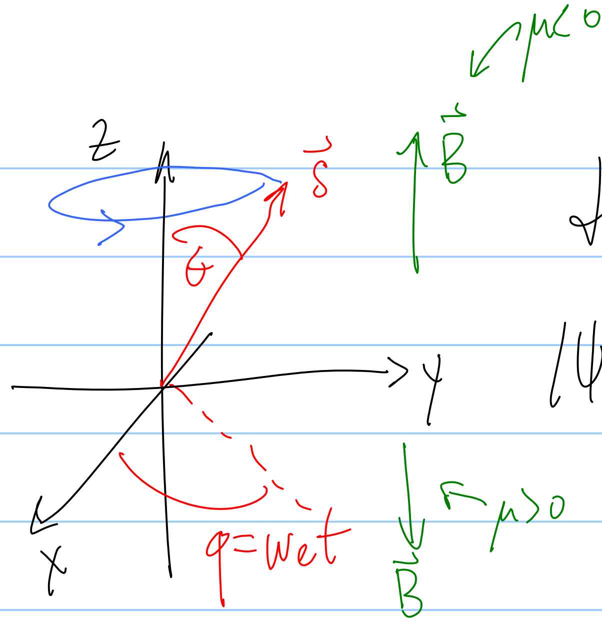
picture drawn w/ $\gamma > 0$

$\omega_0 = |\gamma||B_0|$ Larmor frequency!

For the equivalent spin-1/2 particle:

start with:

$$|\psi(t=0)\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle$$



Then, at a later time:

$$|\psi(t)\rangle = \cos\frac{\theta}{2}\, e^{-i\omega_e t/2}\,|+\rangle + \sin\theta/2\, e^{i\omega_e t/2}\,|-\rangle$$

$$\langle S_z\rangle = \frac{\hbar}{2}\cos\theta$$

$$\langle S_x\rangle = \frac{\hbar}{2}\sin\theta\cos(\omega_e t)$$

$$\langle S_y\rangle = \hbar/2\,\sin\theta\,\sin(\omega_e t)$$

<u>What if we apply an oscillating transverse field?</u>

$$\vec{B}_1 = B_1\cos\omega t\,\hat{x}$$

- now, we solve this by going to a frame rotating at $\vec{\Omega}$
- recall that $\dot{\vec{L}}' = \dot{\vec{L}} - \vec{\Omega}\times\vec{L}'$ in the rotating frame

☆ We will assume $\gamma < 0$ in what follows

- without the transverse field, take rotating frame with $\vec{\Omega} = \omega_0 \hat{z}$ then $B_{eff} = 0$ and the spin vector is stationary
- with the transverse field, take $\vec{\Omega} = \omega \hat{z}$

let's rewrite $\vec{B}_1$: $\vec{B}_1 = \overbrace{\frac{B_1}{2}\left(\underbrace{\cos\omega t\,\hat{x} + \sin\omega t\,\hat{y}}_{\hat{x}'}\right)}^{\text{co-rotating}} + \underbrace{\frac{B_1}{2}\left(\cos\omega t\,\hat{x} - \sin\omega t\,\hat{y}\right)}_{\text{counter-rotating}}$

- let's say that B_1 is small compared with B_0, and $|\omega - \omega_0| \ll \omega_0$

- then, in the rotating frame, $\vec{L}$ and the co-rotating term move together, and the co-rotating torque has a big effect — the counter-rotating torque is whizzing around at 2ω and constantly reversing direction, and can't have much effect

RWA – rotating wave approximation – ignore counter-rotating term – this is not always so good in atomic systems, so watch out!

in the rotating frame, $\vec{B}_1 = \frac{B_1}{2}\hat{x}'$

and $\dot{\vec{L}}' = \gamma \vec{L} \times \left(B_0\hat{z} + \frac{B_1}{2}\hat{x}'\right) - \omega\hat{z}\times\vec{L}$

$$= \vec{L}\times\left(\gamma B_0\hat{z} + \gamma\frac{B_1}{2}\hat{x}' + \omega\hat{z}\right)$$

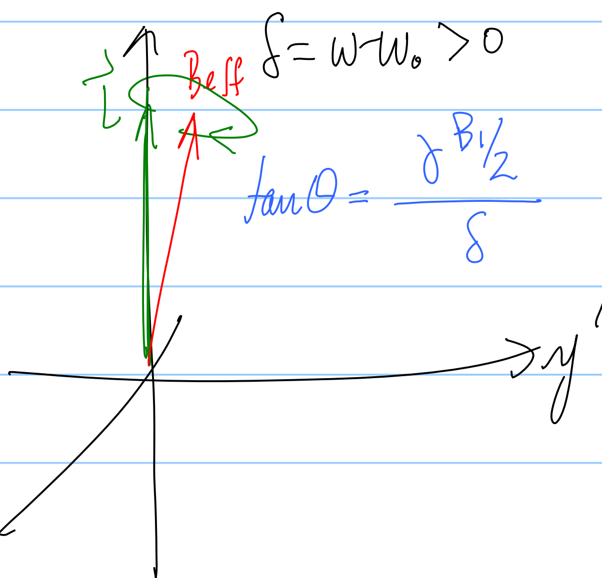
$\delta = \omega - \omega_0$ = detuning

$$\dot{\vec{L}}' = \vec{L}\times\underbrace{\left[\gamma B_1/2\,\hat{x}' + (\overbrace{\omega - \omega_0}^{\delta})\hat{z}\right]}_{\text{effective magnetic field} = \vec{B}_{eff}}$$

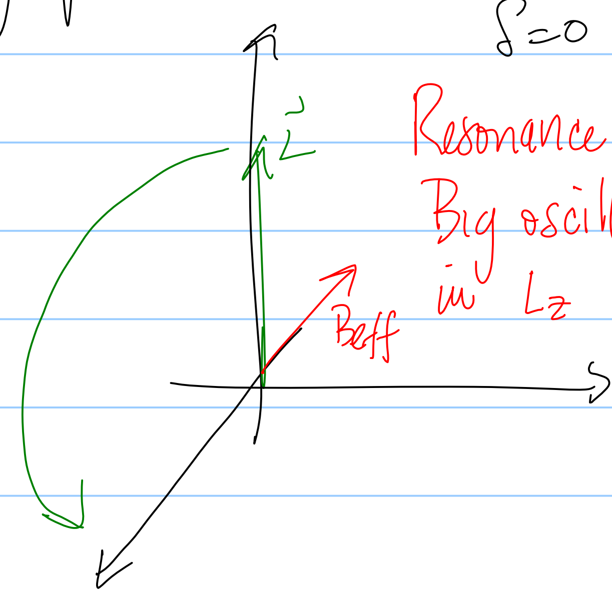
So, what happens in the rotating frame?

$\delta = \omega - \omega_0 > 0$

$\tan\theta = \dfrac{\gamma B_1/2}{\delta}$



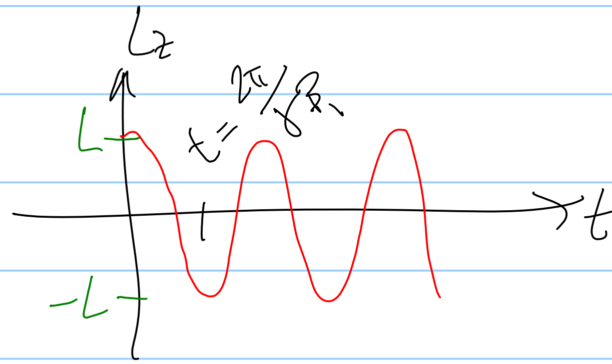
$\delta = 0$



Resonance!
Big oscillations in L_z

This picture is useful! Just have to keep track of the "spin" and "torque"...

Consider resonance:



$\vec{L}(t=0) = L\hat{z}$

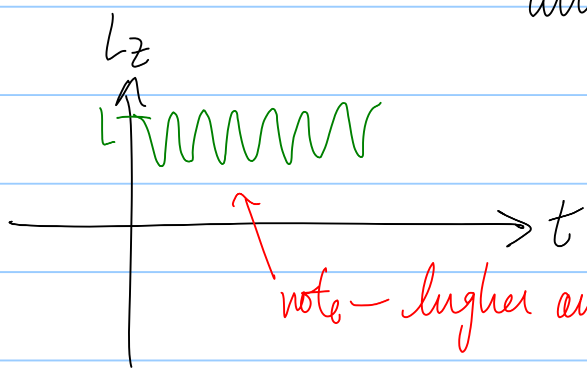
$\dot{\vec{L}} = \vec{L} \times \frac{\gamma B_1}{2} \hat{x}'$

$\frac{|\dot{\vec{L}}|}{L} = \frac{|\gamma B_1|}{2}$ $\quad$ $\vec{L}$ always in $y'-z$ plane and rotates around x' with angular speed $\frac{|\gamma B_1|}{2}$

So, after $t = \frac{2\pi}{\gamma B_1}$, $\vec{L} = L(-\hat{z})$

This is a "π-pulse"!

<u>If we're defined:</u> $\vec{L}$ swings around B_{eff} at angular rate $\sqrt{\left(\frac{\gamma B_1}{2}\right)^2 + \delta^2}$



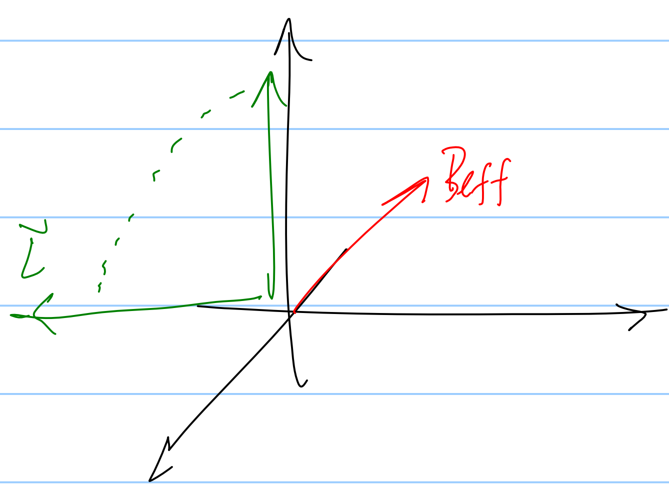
note — higher angular frequency, less "contrast"

<u>Also:</u>

If $B_1 = B_1(\cos\omega t\,\hat{x} + \sin\omega t\,\hat{y})$, then the torque is twice as large (counter-rotating part is not wasted)

Consider applying a $\pi/2$-pulse using:

$$\vec{B}_1 = B_1(\cos\omega_0 t\,\hat{x} + \sin\omega_0 t\,\hat{y})$$



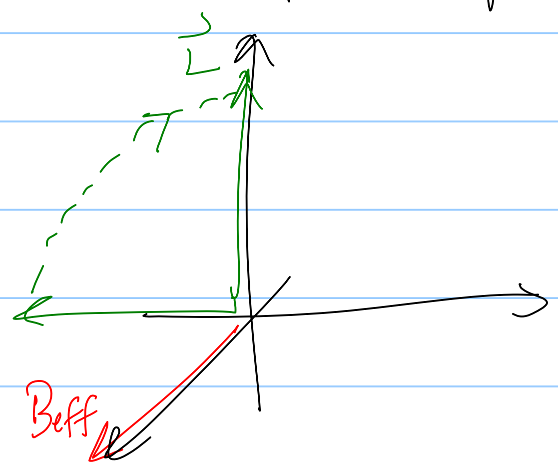
<u>An aside</u>: This is the state $\frac{1}{\sqrt{2}}(|+\rangle - i|-\rangle)$ in the equivalent quantum problem

Now, change the phase of the drive:

$$\vec{B}_1 = B_1 \left[\cos(\omega_0 t - \pi)\,\hat{x} - \sin(\omega_0 t - \pi)\,\hat{y} \right]$$

In the rotating frame, $\vec{B}_1 = \vec{B}_1 (-\hat{x}')$

Apply $\pi/2$-pulse again:



Back where we started!

By playing with the relative phase between pulses, we can get the Bloch vector to go anywhere!

This is also how we can make any quantum superposition state

It should be clear now that the action of the driving field can be represented by a rotation:

$$\mathcal{D} = e^{-i\vec{\sigma}\cdot\hat{n}\,\alpha/2}$$

$\hat{n}$ determined by relative phase and detuning and

$\alpha = \sqrt{\left(\frac{\gamma B_1}{2}\right)^2 + \delta^2} \cdot t$

i.e. • with $\delta = 0$, $\hat{n}$ lies in x-y plane with direction determined by φ_1

• with $\delta = \frac{|\gamma| B_1}{2}$, $\hat{n} = 1/\sqrt{2}\,(\hat{z} - \hat{x})$

After interacting with the driving field, the state ket is just transformed according to the rotation operator

$$|\psi\rangle \longrightarrow e^{-i\vec{\sigma}\cdot\hat{n}\,\alpha/2}\,|\psi\rangle$$